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THE STABILITY OF REGULAR GRIOLI PRECESSIONS†

V. N. TKHAI

Moscow

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In the plane of two essential parameters of the problem, the region the necessary and (individually) sufficient conditions of stability of regular Grioli precessions in the problem of the motion of a heavy, dynamically asymmetrical rigid body about a fixed point is defined. The existence of periodic motions close to regular Grioli precessions is also proved. A method that requires only in solutions of the Cauchy problem to calculate the characteristic exponents of a reversible linear periodic system of order $l + n$ ($l \geqslant n$) is proposed, © 2001 Elsevier Science Ltd. All rights reserved.

1. THE STATEMENT OF THE PROBLEM

In 1947, Grioli [1] discovered regular precessions of a heavy rigid body anchored at such a point that the following conditions are satisfied

$$x_0\sqrt{B-C} = z_0\sqrt{A-B}, \quad y_0 = 0, \quad A > B > C \tag{1.1}$$

where x_0, y_0, z_0 are the coordinates of the centre of gravity G of the body in a moving coordinate system $Oxyz$, with axes directed along the axes of inertia of the body for the fixed point O, and A, B and C are the principal moments of inertia of the body about these axes. Two remarkable features distinguish Grioli precessions. First they occur in a dynamically asymmetrical rigid body subject to conditions (1.1) only. Secondly, Grioli precessions do not occur about a vertical axis but about an axis inclined to the vertical at a certain angle β (Fig. 1).

$$\cos\beta = \frac{A-B+C}{\sqrt{(A-B)(B-C)+(A-B+C)^2}} \tag{1.2}$$

Here, the precession velocity ω_p is equal to the velocity of the eigen rotation of the body ω_0, and the angle between the axes of these rotations is a right angle. Furthermore, the axis of the eigen rotation passes through the centre of mass of the body. In Fig. 1, the ascending vertical γ at the fixed point O, the instantaneous angular velocity of the body ω and also the vectors ω_p and ω_0 are depicted at the instant of time when they belong to the same vertical plane II (the plane of the drawing).

It can be seen from formula (1.2) that regular Grioli precessions are also possible for bodies for which the right-hand side of relation (1.2) is less than $1/\sqrt{2}$. In this case, the angle $\beta > \pi/4$, and in Grioli motions the vector ω spends the entire time on one side of the vertical plane that passes through the fixed point and is perpendicular to the plane II. For such bodies, the moments of inertia should obviously satisfy the condition

$$(A-B)(B-C) > (A-B+C)^2 \tag{1.3}$$

In 1944, Gulyayev [2] found explicit formulae that describe the Grioli solutions

$$p = \frac{n}{l}(x_0 - z_0\cos\tau), \quad q = n\sin\tau, \quad r = \frac{n}{l}(z_0 + x_0\cos\tau)$$

$$\begin{aligned} \gamma_1 &= -\frac{n^2}{Pl^2}[Cz_0\cos\tau + (B-C)x_0\sin^2\tau] \\ \gamma_2 &= \frac{n^2}{Pl^3}[(Ax_0^2 + Cz_0^2) - (A-C)x_0z_0\cos\tau]\sin\tau \end{aligned} \tag{1.4}$$

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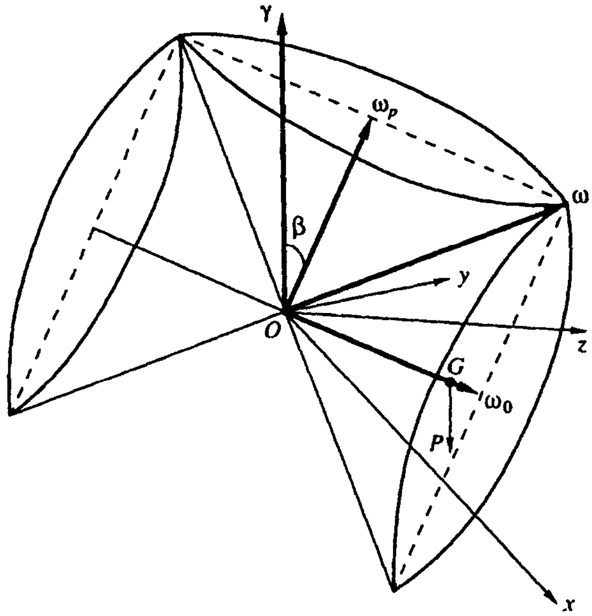

Fig. 1

$$\gamma_3 = \frac{n^2}{Pl^2}[Ax_0 \cos\tau + (A-B)z_0 \sin^2\tau]$$

$$l^2 = x_0^2 + z_0^2, \quad \tau = nt - \varepsilon + \pi/2, \quad \varepsilon = nt_0, \quad n^2 = P^2$$

where p, q and r are projections of the instantaneous angular velocity ω onto the axes of the $Oxyz$ system γ_1, γ_2 and γ_3 are the direction cosines of the vector γ, $P = mg$ is the body weight, l is the distance from the fixed point O to the centre of mass G, n is the angular velocity of the eigen rotation, equal to the angular velocity of precession, and t_0 is the time constant. It can be seen that, for a specified body with a fixed point satisfying the anchoring conditions (1.1), mechanically unique motion is possible in the form of a precession, described by formulae (1.4).

When obtaining explicit expressions (1.4) we can correctly formulate the problem of the stability of regular Grioli precessions as the problem of the stability of the particular solution (1.4) of the system of Euler–Poisson equations describing the motion

$$A\frac{dp}{dt} - (B-C)qr = P(z_0\gamma_2 - y_0\gamma_3), \quad \frac{d\gamma_1}{dt} = \gamma_2 r - \gamma_3 q$$
$$(pqr, ABC, x_0y_0z_0, \gamma_1\gamma_2\gamma_3) \tag{1.5}$$

when conditions (1.1) are satisfied. However, the problem is fairly complex. In fact, first, the equations of perturbed motion will be 2π-periodic in τ, and an investigation even of the linear approximation is not feasible analytically. Second, we have a fairly singular problem in which the linear approximation contains a fourfold zero characteristic exponent with three groups of solutions. This follows directly from the first integrals of the Euler–Poisson equations – the energy and angular momentum integrals and the geometric integral

$$Ap^2 + Bq^2 + Cr^2 + 2P(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = 2h(\text{const})$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \sigma(\text{const}) \tag{1.6}$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

Furthermore, the presence of these zero exponents indicates the instability of the system of equations in variations, and it complicates the problem of the stability of regular Grioli precessions still further.

On the other hand, integrals (1.6) enable as in principle, to reduce the Euler–Poisson equations to a second-order system or to a single second-order differential equation. This system or equation is attractive in its low dimensionality and contains no zero characteristic exponents governed by integrals (1.6). However, as a result, very complex equations are obtained [3, 4], and, furthermore, for these equations it is difficult to obtain explicit formulae describing the Grioli solutions.

The circumstances indicated above lead to the idea that the solution of the problem of the stability of regular Grioli precessions must be sought by analysing the fairly simple system of equations (1.5), but bearing in mind in this case the conceivable possibility of reducing Eqs (1.5) to a second-order system. Another idea is to take into account the properties of reversibility possessed by system (1.3) when $y_0 = 0$. Strictly speaking, the system of Euler–Poisson equations is reversible when the body is anchored at any point O. This is expressed in the invariance of system (1.5) under the replacement of (t, γ, ω) by $(-t, \gamma, -\omega)$. The property of reversibility is useful when examining periodic motions. For example, the periodicity of motion in which the angular velocity vanishes at least twice follows directly from the reversibility of the system of Euler–Poisson equations.

When $y_0 = 0$, system (1.5) is also invariant under the replacement of $(t, \gamma_1, \gamma_2, \gamma_3, p, q, r)$ by $(-t, \gamma_1, -\gamma_2, \gamma_3, p, -q, r)$. In other words, when $y_0 = 0$, system (1.5) is a reversible mechanical system with the fixed point set $M = \{\gamma_1, \gamma_2, \gamma_3, p, q, r\colon \gamma_2 = q = 0\}$, and precessions (1.4) constitute periodic motion that is symmetrical with respect to the fixed-point set M. Therefore, it is necessary not only to reduce the order of system (1.5) to two using integrals (1.6) but also to obtain a reduced periodic second-order system possessing the property of reversibility. It is then possible to use the well-known assertion [5] concerning Lyapunov stability, which nearly always occurs in the case of pure imaginary characteristic exponents, and the characteristic exponents can be calculated by analysing system (1.5) in the neighbourhood of solution (1.4).

2. THE REDUCED SYSTEM

The geometric integral will be taken into account by replacing the variables

$$\gamma_1 = \sin\theta\cos\varphi, \quad \gamma_2 = \sin\theta\sin\varphi, \quad \gamma_3 = \cos\theta \quad (0 < \theta \leqslant \pi) \tag{2.1}$$

Then, instead of three equations for γ_1, γ_2 and γ_3, we obtain only two equations

$$\theta^\cdot = -\omega_2, \quad \varphi^\cdot = -\omega_3 + \omega_1 \operatorname{ctg}\theta \quad (\sin\theta \neq 0)$$

$$\omega_1 = p\cos\varphi + q\sin\varphi, \quad \omega_2 = -p\sin\varphi + q\cos\varphi, \quad \omega_3 = r$$

Note that the angle θ is the angle of nutation, while the angle φ in sum with the angle of eigen rotation gives $\pi/2$. Furthermore, ω_1 and ω_2 are the projections of the angular velocity ω onto axes rotated in the xy plane with respect to the x and y axes by an angle φ.

We will transform the energy integral

$$(A\cos^2\varphi + B\sin^2\varphi)\omega_1^2 + (A\sin^2\varphi + B\cos^2\varphi)\omega_2^2 + (B - A)\sin 2\varphi\omega_1\omega_2 + C\omega_3^2 +$$
$$+2P(x_0\sin\theta\cos\varphi + y_0\sin\theta\sin\varphi + z_0\cos\theta) = 2h$$

and the angular momentum integral to new-variables. From the latter we find

$$\omega_1 = \frac{\sigma - (B - A)\sin 2\varphi\sin\theta\omega_2/2 - C\omega_3\cos\theta}{(A\cos^2\varphi + B\sin^2\varphi)\sin\theta} \tag{2.2}$$

and we substitute it into the energy integral. As a result, we obtain a quadratic equation for determining ω_3, from which we find

$$\omega_3 = F \pm \sqrt{F^2 - SR} \tag{2.3}$$

$$F = C\sigma\cos\theta, \quad S = C[(A\cos^2\varphi + B\sin^2\varphi)\sin^2\theta + C\cos^2\theta]$$

$$R = \sigma^2 - (B - A)^2\sin^2\varphi\cos^2\varphi\sin^2\theta\omega_2^2 + 2[P(x_0\sin\theta\cos\varphi + y_0\sin\varphi + z_0\cos\theta) - h]\times$$
$$\times(A\cos^2\varphi + B\sin^2\varphi)\sin^2\theta$$

As a result of these calculations, the projections ω_1 and ω_2 are expressed by means of formulae (2.2) and (2.3) as functions of φ, θ and ω_2. Therefore, the reduced third-order system has the form

$$\frac{d\theta}{dt}=-\omega_2,\quad \frac{d\omega_2}{dt}=\Omega_2(\varphi,\theta,\omega_2),\quad \frac{d\varphi}{dt}=-\omega_3(\varphi,\theta,\omega_2)+\omega_1(\varphi,\theta,\omega_2)\operatorname{ctg}\theta \tag{2.4}$$

$$\Omega_2\equiv\Big[-\omega_1+\Big(\frac{C-B}{A}q\sin\varphi+\frac{C-A}{B}p\cos\varphi\Big)\omega_3-$$
$$-\frac{P}{A}(z_0\sin\theta\sin\varphi-y_0\cos\theta)\sin\varphi+\frac{P}{B}(x_0\cos\theta-z_0\sin\theta\cos\varphi)\cos\varphi\Big]_*$$

(the asterisk denotes that, on the right-hand side p, q, ω_1 and ω_3 are replaced by their expressions in terms of φ, θ, ω_2). Here, the right-hand sides of system (2.4) will be 2π-periodic in the angle φ and, furthermore, depend on the constants of energy, h, and kinetic moment, σ.

System (2.4) obviously describes motions in which $\sin\theta\neq 0$.

We will now consider regular Grioli precessions. From formulae (1.4) it can be seen that $\gamma_1^2+\gamma_2^2\neq 0$. In fact, when $\gamma_1^2+\gamma_2^2=0$, it is necessary to have $\sin\tau\neq 0$ and $\cos\tau\neq 0$. However, the expressions in square brackets in the formulae for γ_1 and γ_2 cannot then vanish simultaneously, since $x_0z_0>0$ and $(A-B)(B-C)>0$.

Thus, on regular Grioli precessions we have $\sin\theta\neq 0$. It is obvious that $\sin\theta\neq 0$ is also observed in motions close to regular Grioli precessions, i.e. in perturbed motions.

We will calculate

$$\gamma_2^{\boldsymbol{\cdot}}\gamma_1-\gamma_2\gamma_1^{\boldsymbol{\cdot}}=-n^2[f_1(\tau)+f_2(\tau)]/(P^2l^5) \tag{2.5}$$
$$f_1(\tau)\equiv[(Ax_0^2+Cz_0^2)z_0-(B-C)l^2x_0\cos\tau]C$$
$$f_2(\tau)\equiv(A-C)(B-C)x_0^2(z_0-x_0\cos\tau)\sin^2\tau$$

When $|z_0|>|x_0|$, the sign of $f_2(\tau)$ is the same as the sign of the number z_0, but, when $z_0=x_0$, $f_2(\tau)$ vanishes only when $\sin\tau=0$. Furthermore, conditions (1.1) enable as to transform $f_1(\tau)$ into

$$f_1(\tau)=Cz_0f^*(\tau),\quad f^*(\tau)\equiv(A-B+C)-\sqrt{(A-B)(B-C)}\cos\tau$$

and $f_1(\tau)$ has the sign of the number z_0 if

$$A-B+C>\sqrt{(A-B)(B-C)} \tag{2.6}$$

Inequality (2.6) is exactly opposite to inequality (1.3). Therefore, geometrically condition (2.6) means that the angle between the vertical and the axis of precession does not exceed $\pi/4$. We will show here, however, that, when $x_0=z_0$, we have $A+C=2B$ and condition (2.6) is satisfied.

Thus, on regular Grioli precessions, for which the angle formed by the axis of precession with the vertical does not exceed $\pi/4$ and, furthermore, $|z_0|>|x_0|$, the angle φ changes monotonically, as follows from (2.5). Consequently, when solving the problem of the stability of such precessions, it is possible to use a second-order periodic system. This system is derived directly from (2.4) if the angle φ is selected as the new "time".

Let the second-order system obtained be stable with respect to the variables θ and ω_2. In this system, the perturbed and the unperturbed solutions are compared for the same values of φ, which in the general case are reached at different instants of time. Therefore, from the formulae for the transition to the variables φ, θ, ω_1, ω_2 and ω_3 it follows that the stability of the reduced second-order system means the stability of system (1.5) with respect to the variables $\gamma_1^2+\gamma_2^2$, γ_3, p^2+q^2 and r.

Formulae (2.1), and also the expressions for ω_1, ω_2 and ω_3, indicate that, when the sign of γ_2 and q changes, the sign of the angle φ and of the projection ω_2 also changes. In this case, as can be seen from (2.4), the expression for Ω_2 retains its sign. Therefore, system (2.4) is invariant under to the replacement of $(t, \varphi, \theta, \omega_2)$ by $(-t, -\varphi, \theta, -\omega_2)$. This means that the second-order periodic system obtained from (2.4) does not change when $(\varphi, \theta, \omega_2)$ is replaced by $(-\varphi, \theta, -\omega_2)$ and it is a reversible system.

We will now examine Grioli precessions for bodies for which $|x_0|>|z_0|$. In this case, we have $A-B>B-C$, and condition (2.6) is satisfied. Therefore, on Grioli precessions, the angle $\beta<\pi/4$.

Then, as in the previous case, from formulae (1.4) we derive $\gamma_2^2 + \gamma_3^2 \neq 0$. Therefore, if the transformation

$$\gamma_1 = \cos\theta, \quad \gamma_2 = \sin\theta\sin\varphi, \quad \gamma_3 = \sin\theta\cos\varphi$$

is carried out, then, on Grioli precessions, we will have $\sin\theta \neq 0$. Now we will calculate

$$\begin{gathered}\gamma_2^{\bullet}\gamma_3 - \gamma_2\gamma_3^{\bullet} = n^2[f_3(\tau) + f_4(\tau)]/(P^2 l^5) \\ f_3(\tau) \equiv [Ax_0^2 + Cz_0^2)x_0 - (A-B)l^2 z_0\cos\tau]A \\ f_4(\tau) \equiv (A-C)(B-C)z_0^2(x_0 - z_0\cos\tau)\sin^2\tau\end{gathered} \tag{2.7}$$

and here $f_3(\tau)$ is transformed into $Ax_0 f^*(\tau)$. Therefore, when $|x_0| > |z_0|$, expression (2.7) has the same sign and the angle φ changes monotonically.

We will derive equations for the variables

$$\theta, \varphi, \omega_1 = p, \quad \omega_2 = -r\sin\varphi + q\cos\varphi, \quad \omega_3 = r\cos\varphi + q\sin\varphi$$

and, using the energy and angular momentum integrals, we will reduce the system obtained to a third-order system for the variables θ, φ and ω_2. Finally, taking the inequality $\varphi^* \neq 0$ into account, we will obtain a periodic reversible second-order system. These calculations were carried out in detail above for the case when $|z_0| \geqslant |x_0|$.

We will now summarise the conclusions of this section.

Lemma 1. The problem of the stability of regular Grioli precessions, in which the angle which the axis of precession makes with the vertical does not exceed $\pi/4$, correctly reduces to the problem of the stability of a periodic reversible second-order system of the form (2.4). Here, the Lyapunov stability of this system means the stability of system (1.5) with respect to the variables $\gamma_1^2 + \gamma_2^2$, γ_3, $p^2 + q^2$ and r when $|z_0| \geqslant |x_0|$ and with respect to the variables γ_1, $\gamma_2^2 + \gamma_3^2$, p and $q^2 + r^2$ when $|x_0| \geqslant |z_0|$, provided the constants of the energy and angular momentum integrals are not disturbed.

3. CALCULATION OF THE CHARACTERISTIC EXPONENTS OF THE REVERSIBLE SYSTEM

Let us return to the problem of the characteristic exponents (CEs) for a linear, 2π-periodic reversible system

$$\begin{gathered}\mathbf{u}^{\bullet} = \mathbf{A}_-(t)\mathbf{u} + \mathbf{A}_+(t)\mathbf{v} \\ \mathbf{v}^{\bullet} = \mathbf{B}_+(t)\mathbf{u} + \mathbf{B}_-(t)\mathbf{v}, \quad \mathbf{u} \in \mathbb{R}^l, \quad \mathbf{v} \in \mathbb{R}^n (l \geqslant n)\end{gathered} \tag{3.1}$$

(the plus or minus sign denotes a matrix consisting of even or odd functions). It is clear that system (3.1) is invariant under each of the two transformations: a) $(t, \mathbf{u}, \mathbf{v}) \to (-t, \mathbf{u}, -\mathbf{v})$; b) $(t, \mathbf{u}, \mathbf{v}) \to (-t, -\mathbf{u}, \mathbf{v})$. The sets $\mathbf{M}_1 = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ and $\mathbf{M}_2 = \{\mathbf{u}, \mathbf{v} : \mathbf{u} = \mathbf{0}\}$ $(\mathbf{u} = \mathbf{0})$ are referred to as fired-point sets of reversible system (3.1).

The property of invariance under these transformations leads to the fact that system (3.1), together with each solution $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{v} = \mathbf{v}(t)$, also has the solutions $\mathbf{u} = \mathbf{u}(-t)$, $(\mathbf{v} = -\mathbf{v}(-t)$ and $\mathbf{u} = -\mathbf{u}(-t)$, $\mathbf{v} = -\mathbf{v}(-t)$. The reflexivity of the characteristic equation for system (3.1) is derived directly from this. The linearity of system (3.1) implies that the following solutions exist

$$\begin{gathered}\mathbf{u} = \mathbf{u}(t) + \mathbf{u}(-t), \quad \mathbf{v} = \mathbf{v}(t) - \mathbf{v}(-t) \\ \mathbf{u} = \mathbf{u}(t) - \mathbf{u}(-t), \quad \mathbf{v} = \mathbf{v}(t) + \mathbf{v}(-t)\end{gathered} \tag{3.2}$$

In the first of these solutions we have $\mathbf{u} = \mathbf{0}$ when $t = 0$, and in the second solution $-\mathbf{v} = \mathbf{0}$ when $t = 0$. Solutions (3.2) are symmetrical with respect to the sets $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively.

When the condition of uniqueness of the solution of system (3.1) is satisfied, taking solutions (3.2) into account we derive the fundamental matrix of solutions

$$S(t) = \begin{Vmatrix} \mathbf{u}^+(t) & \mathbf{u}^-(t) \\ \mathbf{v}^-(t) & \mathbf{v}^+(t) \end{Vmatrix}, \quad S(0) = \mathbf{I}_{l+n}$$

(the matrices $\mathbf{u}^+(t)$ and $\mathbf{v}^+(t)$ consist of even functions, the matrices $\mathbf{u}^-(t)$ and $\mathbf{v}^-(t)$ consist of odd functions and $\mathbf{I}_j$ is the identity j-matrix) with the identity matrix of the initial conditions.

The rank of the matrix $\mathbf{v}^-(t)$ does not exceed n. This means that, as a result of elementary transformation, l–n columns of the matrix $\mathbf{v}^-(\pi)$ become zero. Then, from the condition rank $\mathbf{v}^-(\pi) \leqslant n$ it follows [6] that in (3.1) at least l–n 2π-periodic solutions exist that are symmetrical with respect to the set $\mathbf{M}_1$.

Thus, the periodic reversible system (3.1) has at least l-n simple zero CEs. The remaining CEs are divided into pairs $\pm \varkappa$.

We will use ρ to denote the eigenvalue of the matrix $S(2\pi)$. Then, the matrices $S(-2\pi) = S^{-1}(2\pi)$ and $2S = S(2\pi) + S(-2\pi)$ have eigenvalues equal respectively to ρ^{-1} and $2\alpha = \rho + \rho^{-1}$. The explicit form of the matrices $S(2\pi)$ and $S(-2\pi)$ enable as to calculate

$$\det(\mathbf{S} - \alpha\mathbf{I}_{l+n}) = \det(\mathbf{u}^+(2\pi) - \alpha\mathbf{I}_l)\det(\mathbf{v}^+(2\pi) - \alpha\mathbf{I}_n) \tag{3.3}$$

On the other hand

$$\det(\mathbf{S} - \alpha\mathbf{I}_{l+n}) = c(\alpha - 1)^{l-n} P_n^2(\alpha), \quad c = \text{const} \tag{3.4}$$

(where $P_n(\alpha)$ is the nth polynomial in α), or the matrix S has at least l–n eigenvalues, equal to unity, and each of the remaining eigenvalues has an even multiplicity. In the case of simple roots of the polynomial

$$Q_n(\alpha) = \det(\mathbf{v}^+(2\pi) - \alpha\mathbf{I}_n)$$

from (3.3) and (3.4) we directly derive

$$Q_n(\alpha) = c_1 P_n(\alpha), \quad c_1 = \text{const} \tag{3.5}$$

However, if $Q_n(\alpha)$ has a root of multiplicity k, then S has an eigenvalue of multiplicity $2k$, and $P_n(\alpha)$ has root of multiplicity k. Consequently, in this case we also have (3.5).

Lemma 2. The linear, 2π-periodic reversible system (3.1) has at least $l - n$ simple zero CEs. All the remaining CEs are divided into pairs $\pm \varkappa$ and are defined by the formulae

$$\varkappa = \frac{1}{2\pi}\operatorname{Arch}\alpha, \quad \det(\mathbf{v}^+(2\pi) - \alpha\mathbf{I}_n) = 0$$

Corollary. If the roots of the polynomial $Q_n(\alpha)$ are real and do not exceed unity in modulus, then all the CEs have zero real parts.

Remark. The following conclusion can be drawn from Lemma 2: to calculate the CEs of a reversible linear periodic $(l + n)$th-order system $(l \geqslant n)$, it is sufficient to construct only n particular solutions of the Cauchy problem in the segment $[0, 2\pi]$.

4. ANALYSIS OF THE SYSTEM OF EQUATIONS IN VARIATIONS

We will change in the Euler–Poisson equations to dimensionless projections of the angular velocity $p^* = p/n$, $q^* = q/n$ and $r^* = r/n$ and introduce the dimensionless parameters

$$a = A/(A+C), \quad c = 1 - a, \quad b = a + (1-2a)\lambda^2, \quad \lambda = x_0/l, \quad \nu = z_0/l, \quad \mu = \sqrt{a^2\lambda^2 + c^2\nu^2}$$

Then, in the system obtained, the Grioli precessions are given by the formulae

$$p^* = \lambda - \nu\cos\tau, \quad q^* = \sin\tau, \quad r^* = \nu + \lambda\cos\tau$$

$$\begin{aligned}\gamma_1 &= -[c\nu\cos\tau + (b-c)\lambda\sin^2\tau]/\mu\\ \gamma_2 &= [(a\lambda^2 + c\nu^2) - (a-c)\lambda\nu\cos\tau]\sin\tau/\mu\\ \gamma_3 &= [a\lambda\cos\tau + (a-c)\nu\sin^2\tau]/\mu\end{aligned} \tag{4.1}$$

and this system contains only two essential parameters: a and $\lambda(1/2 < a < 1, -1 \leqslant \lambda \leqslant 1)$.

We will now compile equations in variations for solution (4.1), putting $\mathbf{u} = (\delta p^*, \delta r^*, \delta\gamma_1, \delta\gamma_2)^T$ and $\mathbf{v} = (\delta q^*, \delta\gamma_2)^T$, where T stands for transposition. As a result, we have the system

$$\begin{aligned}u_1^\bullet &= (b-c)/a(q^*u_2 + r^*v_1) + (\mu/a)\nu v_2\\ u_2^\bullet &= ((a-b)/c)(q^*u_1 + p^*v_1) - (\mu/c)\lambda v_2\\ u_3^\bullet &= \gamma_2 u_2 - q^*u_4 - \gamma_3 v_1 + r^*v_2\\ u_4^\bullet &= -\gamma_2 u_1 + q^*u_3 + \gamma_1^* v_1 - p^*v_2\\ v_1^\bullet &= ((c-a)/b)(r^*u_1 + p^*u_2) - (\mu/b)(\nu u_3 - \lambda u_4)\\ v_2^\bullet &= \gamma_3 u_1 - \gamma_1 u_4 - r^*u_3 + p^*u_4\end{aligned} \tag{4.2}$$

i.e. a system of the form (3.1) with the vectors $\mathbf{u}$ and $\mathbf{v}$ of dimensionality $l = 4$ and $n = 2$ respectively. Consequently, system (4.2) has at least two simple zero CEs. Moreover, because the Euler–Poisson equations possess three first integrals, system (4.2) has a further pair of zero CEs with the unique group of solutions.

To calculate the remaining pair of CEs, we will use Lemma 2. In the segment $[0, 2\pi]$ we will construct two solutions of the Cauchy problem with the initial conditions: 1) $\mathbf{u}(0) = 0$, $\nu_1(0) = 1$, $\nu_2(0) = 0$ and 2) $\mathbf{u}(0) = 0$, $\nu_1(0) = 0$, $\nu_2(0) = 1$. As a result, we form the matrix $\mathbf{v}^+(2\pi) = \|\nu_{ij}^+(2\pi)\|$. We then determine the roots α of the quadratic equation

$$\alpha^2 - [\upsilon_{11}^+(2\pi) + \upsilon_{22}^+(2\pi)]\alpha + \upsilon_{11}^+(2\pi)\upsilon_{22}^+(2\pi) - \upsilon_{12}^+(2\pi)\upsilon_{21}^+(2\pi) = 0 \tag{4.3}$$

and the CEs $\pm\varkappa$, $\varkappa = \operatorname{Arch}\alpha/(2\pi)$In this case, Eq. (4.3) has one root $\alpha = 1$. Therefore, it is necessary to determine only the second root, equal, by Vistas theorem, to

$$\alpha = \upsilon_{11}^+(2\pi) + \upsilon_{22}^+(2\pi) - 1 \tag{4.4}$$

We find the number α from formula (4.4). Then, the condition $|\alpha| \leqslant 1$ determines the pure imaginary CEs $\pm\varkappa$, which in this case are calculated by the formula $\varkappa = \arccos\alpha/(2\pi)$.

The system of Euler-Poisson equations contains, in the Grioli case, only two essential parameters a and λ, which vary in the rectangle (Fig. 2). For each point of this rectangle, we calculate α and single out the region in which the CEs are pure imaginary. In this case, it is obviously sufficient to carry out calculations only for $\lambda \geqslant 0$ or $\lambda \geqslant 0$. In Fig. 2, the region where the CEs have non-zero real parts is shaded.

5. RESULTS

In the system of equations in variations (4.2), the angle φ, if it is used as the "time", changes in the same way as in Grioli precessions

$$\varphi = \tau + \Phi(\tau), \quad \Phi(\tau + 2\pi) = \Phi(\tau) \tag{5.1}$$

and here $|\Phi^\bullet| < 1$, if the angle β between the axis of precession and the vertical does not exceed $\pi/4$. In the latter case, relation (5.1) enables as to express τ in terms of φ. Therefore, from the pair of pure imaginary CEs of system (4.2) it follows that the unique pair of CEs of the reduced second-order system will also be pure imaginary. In the plane (λ, a), the condition $\beta < \pi/4$ takes the form $(\lambda \geqslant 0)$

$$(1-a)/(2a-1) > \lambda(\sqrt{1-\lambda^2} - \lambda) \tag{5.2}$$

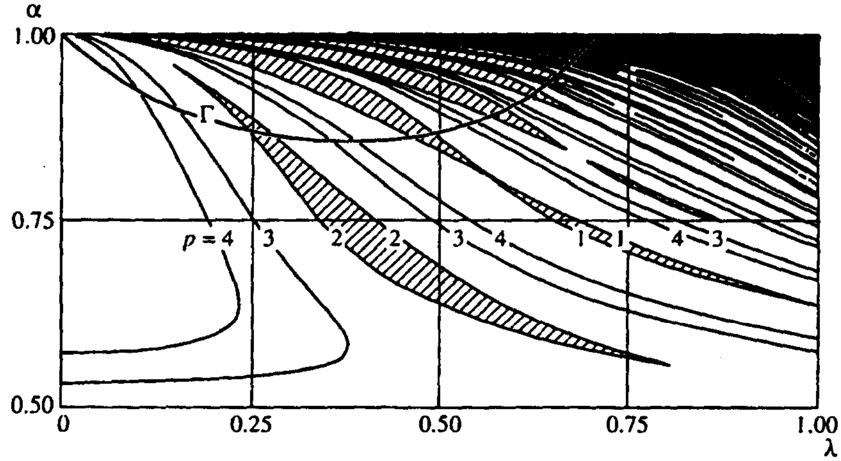

Fig. 2

This condition is satisfied in Fig. 2 in the region D (below the are Γ). In the region D, Lyapunov stability follows from the pure imaginary characteristic exponents (regions S of light background). For this, it is necessary to exclude the resonances $p\varkappa = 1, p = 1, 2, 3, 4$ (the corresponding values of p in Fig. 2 are indicated) and require that the one coefficient $C(\lambda, a)$ does not vanish in the third-order forms of the normalized system (for a system of form (2.4)). The third-order forms in system (2.4) do not vanish identically. In the normalized system, then, we almost always have $C(\lambda, a) \neq 0$.

Theorem 1. Regular Grioli precessions are Lyapunov-unstable for all points of the region IS (shaded in Fig. 2). In the region D lying below the are Γ, the angle between the axis of precession and the vertical does not exceed 45° and the regular Grioli precessions are Lyapunov-stable for almost all points of the region S (light background, Fig. 2). Possible exceptions are the resonance curves, and also the points at which $C(\lambda, a) = 0$.

Lyapunov stability is obtained from the theorem formulated. Even though it would appear that the reduced reversible second-order system contains constants of the energy and angular momentum integrals and the stability exists if and only if their values are fixed (conditional stability). However, a study recently carried out [7] showed that, in the case of symmetrical integrals (such as the energy and angular momentum integrals). Lyapunov stability is obtained.

The calculations carried out in Section 4 to determine the CEs also enable as to solve the problem of the existence in a heavy rigid body with one fixed point of periodic motions close to regular Grioli precessions. Here, two statements of the problem are possible.

Suppose, as before, that conditions (1.1) are satisfied. The Grioli precessions constitute symmetrical periodic motion (SPM) of a reversible mechanical system and, in the general situation, belong to a family [8]. By virtue of the existence of energy and angular momentum integrals with arbitrary (non-fixed) constants, we would expect this to be a two-parameter family. The problem of the existence of a family can be reduced to the problem of the existence of a local family of SPM in the system of equations of perturbed motion. By introducing a small parameter, the latter problem is reduced to the problem of continuation with respect to the parameter and is solved [6] in approximate cases by the system of equations in variations.

In the second problem, we assume that $y_0 = 0$, and that the condition

$$x_0\sqrt{B-C} = z_0\sqrt{A-B} \tag{5.3}$$

is satisfied with some accuracy characterized by the small parameter ε. Then, with $\varepsilon = 0$, we have regular Grioli precessions, but when $\varepsilon \neq 0$ the existence of SPM is shown by continuation with respect to the parameter ε of solutions (1.4). In structurally stable cases, this problem is also solved [6] by the system of equations in variations.

The system of equations in variations (4.2) has a fourfold zero CEs with three groups of solutions. Taking into account that these zero CEs are due to the three first integrals (1.6), we conclude that the zero CEs do not prevent continuation of the solutions (1.4).

Therefore, if the remaining pair $\pm\varkappa$ satisfies the condition $\varkappa \neq ik$ ($k \in \mathbb{Z}$), the problem of continuation has a positive solution [6].

Theorem 2. For almost all values of the parameters, the regular Grioli precessions belong to a two-parameter family of SPM of the system of Euler-Poisson equations. This family exists in the Grioli case (1.1) and extends to the case when condition (5.3) is satisfied with a certain accuracy, and $y_0 = 0$.

The existence of Grioli precessions was established [1, 2] when conditions (1.1) were satisfied. However, a careful reading of [1, 2] shows that the precessions described by formulae (1.4) also exist for bodies subject, instead of conditions (1.1), to the conditions

$$x_0\sqrt{B-C} = -z_0\sqrt{A-B}, \quad y_0 = 0, \quad A > B > C \tag{5.4}$$

which differ from (1.1) in that the signs of x_0 and z_0 are opposite to each other. It can also be seen that the derivation of the results set out above concerning stability and periodic motions remains valid for bodies subject to conditions (5.4). Therefore, the following theorem holds.

Theorem 3. Theorems 1 and 2 remain valid if the body obeys conditions (5.4) rather than conditions (1.1).

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